Interest Rates and Asset Distributions of Naive Hyperbolic Discounters

by

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Abstract

This thesis extends the heterogeneous agent incomplete market model of Huggett (1993) with partially naive hyperbolic discounting. An Euler-equation is derived for the agents with this discounting model. Possible pathologies of this Euler-equation and the consumption function are considered, providing insight into the consequences and side effects of naiveté. The purpose of this paper is the analysis of how parameters of the discounting function influence the risk-free rate and asset distributions. It is found that the level of naiveté plays a particularly important part in determining interest rates – high levels of it imply high interest rates. The other main result is that there might be large inequalities in heterogeneous populations (where heterogeneity is in the hyperbolic discount factor and the level of naiveté). This thesis shows that it also has significant welfare consequences – especially when the heterogeneity is in naiveté. This might be an important area of research in the future for policy analysis.

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Introduction

Can consumption-based asset pricing models have a good empirical fit? This question motivated the development of a large number of models after the seminal paper of Lucas (1978) and the famous critique by Mehra & Prescott (1985). The main purpose of the extensions that came after them was to achieve a better empirical fit, and to provide an explanation for various phenomena that Lucas' original model could not reproduce – chief among them the equity premium puzzle. They were an answer to the question: can we find assumptions such that our models fit the interest rates and returns we observe?

In this thesis the question is the other way around. I take the thoroughly-tested assumption of naive hyperbolic discounting, then find out what happens to the predictions of the Huggett (1993) model (a successor of Lucas' asset pricing model) when this assumption is incorporated. The main questions of this research are: what interest rates do we predict under this more realistic discounting model? How does it influence asset distributions?

This thesis mainly builds on two branches of economics: consumption-based asset pricing with heterogeneous consumers and consumption models with hyperbolic discounting. Its main purpose is to combine these two domains into an interest rate model with behavioral features. It is motivated by the recent trend of behavioral finance – or more broadly behavioral economics – gradually becoming part of mainstream economics. The behavioral part of the model features a discounting model for which there is plenty of evidence and is slowly taking hold even outside of behavioral economics. The finance part is a stripped down consumption-based asset pricing model that is standard and well-known in its field. The combination of the two parts wants to be a step towards bringing back behavioral elements into macroeconomics (Akerlof, 2002).

Chapter 1

Background

1.1 Consumption-based asset pricing

Early Markowitzian portfolio theory was based on the assumption that investors optimize for only one period. Its standard framework, the Capital Asset Pricing Model (CAPM), has a very stylized theoretical structure: portfolio risk is exogenous, it is not linked to macroeconomic variables. Merton (1973) addresses the first issue by introducing an asset pricing model based on intertemporal choice. Based on it the so called Consumption Capital Asset Pricing Model (CCAPM) was derived in the second half of the '70s in a series of papers (Rubinstein, 1976; Breeden, 1979). Its main achievement was linking the discount factor (and thus asset returns) to the marginal utility of consumption. Meanwhile Lucas (1978) introduced a general equilibrium model for asset pricing. Production is exogenous, the representative agent has a stochastic income following a Markov-process, so analysis is very tractable.

Although CCAPM models provide a theoretically sound framework for thinking about asset returns, they do not fit the data very well. One of the more famous phenomena that vanilla CCAPM is unable to reproduce is the so called equity premium puzzle: average equity returns exceed risk-free returns by a much larger margin than plausible levels of risk aversion would imply (Mehra & Prescott, 1985). There has been many attempts to solve this puzzle. Epstein & Zin (1989) invented a recursive utility function which has separate parameters for risk-aversion and the elasticity of intertemporal substitution. Campbell & Cochrane (1995) extended the utility function with habit formation. Habit formation in that model is external, so analysis is relatively simple, and at the same time the model manages to reproduce many empirically observed phenomena. Another approach was proposed by Mehra at the end of his paper on the equity premium puzzle: heterogeneous agents with uninsurable individual risk. This line of models was pioneered by Bewley (1983) and then used for asset pricing by Huggett (1993) and Constantinides & Duffie (1996). These models are made particularly attractive by the fact that having some uninsurable individual risk is an empirically very sound assumption.

The heterogeneous agent approach has since been extended with various features, such as having physical capital (Aiyagari, 1994), overlapping generation (Storesletten *et al.*, 2007), or multiple assets (Longstaff, 2009). However as of the time of writing this thesis there is little literature on incorporating behavioral factors into the Huggett-model. This thesis aims to provide an example of such. Using realistic discounting rules in asset pricing models is instrumental, as discounting is at the hearth of determining interest rates.

1.2 Consumption models with hyperbolic discounting

Despite the lack of consumption-based asset pricing models with behavioral features, there is a sizable literature of consumption models based on hyperbolic discounting. One of the early discussions of this subject is due to Laibson (1994). In the first chapter of his thesis Laibson characterizes the equilibria of the intra-personal consumption game that arise for consumers with time-inconsistent preferences. The later chapters deal with various forms of commitment. Then Laibson (1997) provides a discussion of what happens when a hyperbolic discounter has access to an imperfect commitment device (a golden egg, in the author's terminology).

Since then there has been much research on the consumption choices of hyperbolic discounters. The number of papers written on a simple-looking topic such as the consumption-saving decision (for example Laibson *et al.*, 1998; Angeletos *et al.*, 2001; Harris & Laibson, 2003; Krusell & Smith, 2003) shows well the relevance and complexity of this topic. Most of this research focuses on life-cycle savings, the use of illiquid assets and welfare issues. There are also general equilibrium models with hyperbolic discounters, for example Barro (1999) extends the Ramsey-model with hyperbolic discounting. It looks like hyperbolic discounting is becoming an important part of consumption literature.

Despite this it is a relatively little-used concept for consumption-based asset pricing. There are examples (for example Gong *et al.* (2011) build a representative agent model for the analysis of the equity premium), but as of yet there are no incomplete market heterogeneous agent models with hyperbolic discounters. Also, most of the papers on consumption with hyperbolic discounting feature partial equilibrium models. While they are useful to analyze certain phenomena in an environment isolated from other factors, the results can also be misleading. Furthermore, in most of these consumption models only the fully naive and fully sophisticated cases of hyperbolic discounting are considered (although there are counterexamples; for a consumption model with intermediate levels of naiveté see Skiba & Tobacman, 2008). This thesis aims to fill these niches.

Based on the the evolution of consumption-based asset pricing and consumption models with hyperbolic discounting discussed in this chapter, combining the two for an asset pricing model with hyperbolic discounters seems like a logical next step. This thesis combines two of the simpler models of these two branches (the heterogeneous agent incomplete market economy of Huggett (1993) and the hyperbolic Euler-equation derived by Harris & Laibson (2001)) to accomplish this goal. It also extends the model with intermediate levels of naiveté, because both complete naiveté and full sophistication are relatively strong assumptions.

The main object of analysis are interest rates and asset distributions. Understanding the risk-free rate is important both for itself and for understanding the model. Explaining interest rates is one of the main objectives of finance, but how hyperbolic discounting, time inconsistency and especially naiveté influence them is little-understood. Also in this model the risk-free rate provides the only link between agents, therefore understanding it is key to understand other results of the model, such as inequality. Asset distributions are also interesting to study in this framework. Contrary to usual models with the production side abstracted away, this model is able to produce non-trivial results (especially for heterogeneous populations). At the same time unlike more complex general equilibrium models it remains very tractable, therefore the channels and causes of wealth distribution changes can be identified.

Chapter 2

Hyperbolic discounting

The purpose of this chapter is to motivate using hyperbolic discounting in the Huggett-model. After a brief history of modeling intertemporal choice various examples are shown that refute the traditional exponentional discounting model. After that I present how hyperbolic discounting can offer a solution for these discrepancies, and how naiveté can be incorporated into this model.

2.1 Exponential discounting

For almost 80 years, the canonical model of intertemporal decision-making has been the exponential discounted utility model proposed by Samuelson (1937). The main feature of the model is that the intertemporal choice of the decision maker can be described by one parameter linking the present to the future: the discount factor δ. Even though Samuelson did not think not think of this model as an accurate description of reality, it was quickly adopted (Frederick *et al.*, 2002) and still remains the dominant way of modeling intertemporal choice.

This popularity is due to a number of different reasons, chief of which is that it implies dynamically consistent choices, which makes analysis very tractable. Even Samuelson (1937) shows examples of self-constraining behavior (irrevocable trust funds and life insurance as compulsory savings measures) which would be unnecessary for hyperbolic discounters, and states that the functional form of this model has been selected such that this is not an issue. Samuelson also remarks that "in the analysis of the supply of savings, it is extremely doubtful whether we can learn much from considering such an economic man, whose tastes remain unchanged, who seeks to maximise some functional of consumption alone, in a perfect world, where all things are certain and synchronised".

2.2 Empirical observations

2.2.1 Short-run impatience versus patience about future decisions

The idea, that people are relatively impatient when deciding between present and future consumption, but at the same time relatively patient when making decisions between two dates in the future, is not new. An early mention is due to Strotz (1955), who made the following example: if a choice has to be made between one apple today and two apples tomorrow, some may be tempted to choose the instant gratification from consuming the apple today; however when the choice is between one apple in one year and two apples in one year plus one day, nobody would choose the latter.

Following this logic, the canonical experiment to elicit time preferences is to ask people about how much money should they receive in the future to make them indifferent to receiving a given amount now. Based on such questions Thaler (1981) documented that discount rates seem to vary inversely with the length of time to be waited, whereas exponential discounting would imply one constant discount factor. Since then, there has been a plethora of empirical evidence on both short-run impatience and long-run relative patience.

One of the more famous examples of short-run impatience is the payday effect. It has been observed that household consumption is higher in the days after payday than at the end of the month (Huffman & Barenstein, 2005). An even starker result is that even food expenditure and calorie intake fluctuate similarly over the month (Shapiro, 2005). Another example is short-term borrowing, particularly the so-called payday loans. The high rates on these would imply an implausibly high level of impatience in the hyperbolic discounting model. For example Martin (2010) reports that a typical payday loan at that time allowed a consumer to borrow $400 for fourteen days for a fee of $100, which would imply an APR of over 1,000% and a yearly discount factor below 0.1. In other words, a consumer taking this loan would theoretically reject $100 dollars in a year for $10 today, or more strikingly, she would reject $100,000 in five years for $10 today. Obviously there might reasons for this behavior other than the discount factor (for example unforeseen financial problems), but the fact that these loans are so widespread is a hint that households might not discount their future utilities exponentially. This idea is supported by a study of Skiba & Tobacman (2008), who estimate a structural model of payday borrowing and defaults, and find that they can reject the exponential discounting model.

2.2.2 Time inconsistency

The other type of evidence against exponential discounting is based on the most important feature of the model: time consistency. A consumption plan which is optimal at a given point will be optimal at any point in the future for exponential discounters. One kind of evidence against it was already pointed out by Samuelson: compulsory saving measures, which would be unnecessary in the absence of time inconsistency. More generally, it is easy to come up with examples of people trying to constrain their future behavior. Just to mention one, Ashraf *et al.* (2006) report the success of a financial product designed to allow individuals commit to their savings. This implies that in addition to being subject to time inconsistency regarding their financial decisions, people know about it, and they actively take steps to mitigate its effects.

Another type of evidence is when people systematically mispredict their future choices, and deviate from their plan. It also hints at time inconsistency: the course of action that was optimal when they made their plans is not optimal anymore when they deviate from it. There are plenty of examples of such behavior, for example deciding to join a gym the next month in every month, deciding to start working on the thesis tomorrow for weeks, and so on. An example of this is documented by Read & van Leeuwen (1998). In an experimental setting they find that when office workers are confronted with a choice whether to receive a healthy or an unhealthy snack in the future, most of them choose the healthy one. However given the chance to revise their decision just before getting it, a number of them change their mind and go for the unhealthy snack, but it happens much less frequently the other way around. The main difference between this example and the former one is that in this case those who reverse their decision are ignorant about their time inconsistency, they mispredict their own future preferences.

2.3 Hyperbolic discounting

2.3.1 A model of hyperbolic discounting

Both time consistency and the short-run long-run patience anomaly can be explained by a model in which the discount factor is decreasing over the waiting time. It has been first documented by Herrnstein (1961) that the discount rates of pigeons seem to have a hyperbolic shape when plotted against time delay. Since then a variety of studies confirmed this fact also in human behavior (Ainslie & Haslam, 1992; Green *et al.*, 1994; Paserman, 2008).

In the discounted utility model the utility of an agent at time t is

$$\mathcal{U} = \sum_{\tau=1}^{\infty} D(\tau) u_t$$

where $D(\tau)$ is the discount factor for timd span τ. In the standard exponential discounting model D is an exponential function with one parameter:

$$D(\tau) = \delta^{\tau}.$$

In the hyperbolic discounting model D is a hyperbolic function with two parameters:

$$D(\tau) = (1 + \alpha\tau)^{-\gamma}.$$

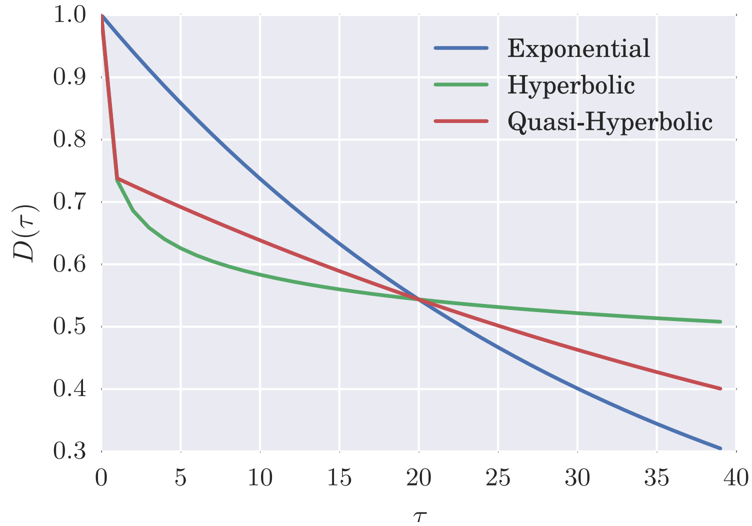
Following Laibson (1997) we can define the instantaneous discount rate as $D'(\tau)/D(\tau)$, which instantly highlights the main difference between hyperbolic and exponential discounting: the discount rate in the exponential model is constant (δ), whereas the discount rate in the hyperbolic model ($\alpha\gamma/(1 + \alpha\tau)$) is decreasing in τ.

In discrete time often the so-called quasi-hyperbolic discount factor is used, which is

$$D(\tau) = \begin{cases} 1 & \text{if } \tau = 0 \\ \beta\delta^{\tau} & \text{if } \tau > 0 \end{cases}$$

with $\beta, \delta \leq 1$. It can be thought of as a generalization of the exponential discounting model (which is a special case of quasi-hyperbolic discounting model with $\beta = 1$). It is also a simpler, more tractable version of the hyperbolic discounting model preserving its main features. First, the agent is relatively impatient between the present and next period (the discount factor is $\beta\delta$) while she is patient between any two consecutive future periods

Figure 2.1: Discount factors



(the discount factor is δ). Second, the quasi-hyperbolic discounting model implies time-inconsistent decisions.

The three different discount factors are shown on figure 2.1. The parameters of the quasi-hyperbolic discount factor are set to plausible values ($\beta = 0.75, \delta = 0.97$), and the parameters of the other discount functions are chosen such that they imply the same discount factor for 20 periods, and the one-period discount factor is the same for the hyperbolic and quasi-hyperbolic case.

For the rest of this thesis, I will use the quasi-hyperbolic discount factor in my model. I will often refer to the quasi-hyperbolic discounting model as hyperbolic discounting, as is usual in the literature.

2.3.2 Sophistication and naiveté

As mentioned earlier, time-inconsistent individuals can behave differently depending on whether they know that they are time-inconsistent or not. The former type of individuals are called sophisticated, whereas the latter are called naive. To illustrate it let us consider the following simple model of decision-making with hyperbolic discounting:

$$\max_{\{x_1,x_2,x_3\}} u(x_0) + \beta\delta u(x_1) + \beta\delta u(x_2)$$
$$\text{s.t. } (x_0, x_1, x_2) \in \Gamma$$

The naive decisionmaker finds the series of x_t which solves the problem, as if future selves would follow through with his plan. Then he chooses x_0 in the current period corresponding to this solution. Then in the next period he solves the problem again, but because of the time-inconsistency of quasi-hyperbolic discounting, he will choose an x_1 which is probably not the same as the x_1 planned by self 0. In other words, he systematically mispredicts his future actions.

The sophisticated decisionmaker understands the time-inconsistency, and when making today's decision, she also considers that her future selves will not necessarily behave according to her plan. She interprets this decision situation as a dynamic game with her future selves, therefore she solves the problem by backwards induction. In this case there is no misprediction of future actions, her beliefs about the future are rational. Additionally in this case if she has access to a commitment device she might use it to constrain her future decisions. For a proper description of sophistication and naiveté with examples see Kőszegi & Rabin (2015).

To generalize the above and allow for intermediate levels of naiveté I will follow O'Donoghue & Rabin (2001). First notice that the naive decisionmaker can be described as a hyperbolic discounter who believes that he will be an exponential discounter from the next period on, whereas the sophisticated decisionmaker can be described as a hyperbolic discounter who knows that she will still be a hyperbolic discounter in the future. Or more succinctly, naifs observe that $\beta = \beta$ in the current period, but believe that $\beta = 1$ in the future, while sophisticates know that $\beta = \beta$ forever. With this insight it is easy to describe intermediate levels of naiveté. Let us denote the level of naiveté by $\widehat{\beta}$ ($\beta \leq \widehat{\beta} \leq 1$). We model this partial naiveté as follows: the decisionmaker observes that $\beta = \beta$ now, but believes that $\beta = \widehat{\beta}$ from the next period on. $\widehat{\beta} = 1$ corresponds to complete naiveté and $\widehat{\beta} = \beta$ means full sophistication.

Chapter 3

Model

In this chapter I introduce a model based on the heterogeneous-agent incomplete-insurance model of Huggett (1993). The main extension is that agents are hyperbolic discounters and are partially naive about it.

3.1 Model setup

Consider an exchange economy with a continuum of agents. In each period each agent receives a stochastic endowment $e \in E$. The endowment follows a Markov-process such that $\pi_{ee'} = \Pr[e_{t+1} = e' \,|\, e_t = e] > 0 \;\forall\, e, e' \in E$. In this application $E = \{e_l, e_h\}$, and the endowments and transition probabilities correspond to the states of employment and unemployment.

Agents are quasi-hyperbolic discounters with parameters $\beta, \delta \leq 1$:

$$\mathcal{U}(c_t, c_{t+1}, c_{t+2}, \dots) = u(c_t) + \mathrm{E}\left[\beta\delta u(c_{t+1}) + \beta\delta^2 u(c_{t+2}) + \dots\right]$$

and u is the CRRA utility function with parameter γ

$$u(c) = \frac{c^{1-\gamma}}{1-\gamma}.$$

Agents are also partially naive with parameter $\widehat{\beta}$.

Agents have access to one non-state-contingent asset: a simple risk-free bond with interest rate r. Consumption must satisfy the period budget balance

$$c_t + \frac{a_{t+1}}{1+r} \leq a_t + e_t$$

and there is also an exogenous borrowing constraint $\underline{a} \leq 0$, so asset holdings must satisfy $a_t \geq \underline{a}$. As $\lim_{0+} u(c) = -\infty$ and $\lim_{0+} u'(c) = \infty$, there is also a natural borrowing constraint $\underline{a}_{nat} = -e_l/r$ which is never reached.

The heterogeneity in the economy is described by Φ, which is a probability distribution over the state space $X = E \times A$ (where $A = [\underline{a}, \infty]$). So for any measurable set $B \subset X$, $\Phi(B)$ is the mass of agents with $(e, a) \in B$. Now given the policy function $a(x)$, a transition function can be defined as $P_a(x, B) = \Pr[x_{t+1} \in B \mid x_t = x]$.

With it everything is set to define the equilibrium in this economy in a similar fashion to Huggett.

Definition (Stationary equilibrium)**.** We call $(c(x), a(x), r, \Phi)$ a stationary equilibrium, if

i) $c(x)$ and $a(x)$ are optimal given r *and beliefs about future* β

ii) Goods and assets markets clear

$$\int_X c(x)\, \mathrm{d}\Phi(x) = \int_X e(x)\, \mathrm{d}\Phi(x)$$
$$\int_X a(x)\, \mathrm{d}\Phi(x) = 0$$

iii) Φ is a stationary distribution given $a(x)$

$$\int_X P_a(x, B)\, \mathrm{d}\Phi = \Phi(B) \text{ for every measurable set } B \subset X$$

$c(x)$ and $a(x)$ are said to be optimal if there is a subgame-perfect equilibrium of the intra-personal game such that the agent chooses $c(x), a(x)$, given the (possibly incorrect) beliefs about future selves' preferences.

3.2 Euler-equation

The derivation and equilibrium concept in the following two chapters are based on the idea of Harris & Laibson (2001). Instead of writing the value function as $V(x_t)$ I will explicitly separate the state variables and use $V_s(a_t)$ instead (where $s \in \{l, h\}$).

First let us assume that the consumption function is continuously differentiable. It is not necessarily true, but it allows for an intuitive derivation of the Euler-equation. Let us denote $R = 1 + r$. Let us further denote the continuation value function by V_s and the *perceived* current value function by W_s in state s. The perceived current value is the value that self t would have in a given situation *if future selves had* $\beta = \widehat{\beta}$. And as self t thinks that the perceived value function is the true value function, she will choose a c_t that maximizes it. This is the key idea that makes it possible to extend Harris & Laibson's model by intermediate levels of naiveté.

By definition the continuation value is

$$V_s(a_{t+1}) = u(c_s(a_{t+1})) + \delta \mathrm{E}_{t+1} V_{s'}(R(a_{t+1} - c_s(a_{t+1})) + e_{s'}). \tag{3.1}$$

When self t makes the consumption decision, she believes that the current value function of self $t+1$ is

$$W_s(a_{t+1}) = u(c_s(a_{t+1})) + \widehat{\beta}\delta \mathrm{E}_{t+1} V_{s'}(R(a_{t+1} - c_s(a_{t+1}) + e_{a'}). \tag{3.2}$$

Thus $V_s(a_{t+1})$ and $W_s(a_{t+1})$ is linked by the following equation according to self t's beliefs:

$$V_s(a_{t+1}) = \frac{1}{\widehat{\beta}} W_s(a_{t+1}) + \frac{1-\widehat{\beta}}{\widehat{\beta}} u(c_s(a_{t+1})) \tag{3.3}$$

At time t the agent maximizes *perceived* current value

$$c_s(a_t) = \underset{c \in \Gamma_s(a_t)}{\arg\max}\, u(c) + \mathrm{E}_t \left[\beta\delta V_{s'}(R(a_t - c) + e_{s'})\right] \tag{3.4}$$

where $\Gamma_s(a) = [0, a - \underline{a} + e_s]$ is the set of feasible consumption choices. The first order condition of the above optimization problem is

$$u'(c_s(a_t)) \geq \mathrm{E}_t \left[R\beta\delta V'_{s'}(R(a_t - c) + e_{s'})\right] \tag{3.5}$$

with equality if $c_s(a_t)$ is in the interior of $\Gamma_s(a_t)$. Now differentiating and substituting in equation 3.3 to 3.5 yields

$$u'(c_s(a_t)) \geq R\frac{\beta}{\widehat{\beta}}\delta \mathrm{E}_t \left[\left(W'_{s'}(a_{t+1}) - (1-\widehat{\beta})u'(c_{s'}(a_{t+1}))\right) c'_{s'}(a_{t+1})\right]. \tag{3.6}$$

Finally we can use the envelope-condition $W'_s(a_{t+1}) = u'(c_s(a_{t+1}))$ to get rid of W'_s in the above inequality. After substituting in and rearranging we get the hyperbolic Euler-equation with naiveté:

$$u'(c_s(a_t)) \geq \mathrm{E}_t \left[R\left(\beta\delta c'_{s'}(a_{t+1}) + \frac{\beta}{\widehat{\beta}}\delta(1 - c'_{s'}(a_{t+1}))\right) u'(c_{s'}(a_{t+1}))\right] \tag{3.7}$$

or if we omit the states for clarity

$$u'(c(a_t)) \geq \mathrm{E}_t \left[R \left(\beta\delta c'(a_{t+1}) + \frac{\beta}{\widehat{\beta}}\delta(1 - c'(a_{t+1})) \right) u'(c(a_{t+1})) \right] \tag{3.8}$$

with equality if $c_t < a_t - \underline{a} + e_t$.

It is easy to see that the standard exponential Euler-equation is a special case of equation 3.7 with $\beta = \widehat{\beta} = 1$, and the strong hyperbolic Euler-equation of Harris & Laibson is also a special case with $\beta = \widehat{\beta} \leq 1$ (full sophistication). The term

$$\beta\delta c'_{s'}(a_{t+1}) + \frac{\beta}{\widehat{\beta}}\delta(1 - c'_{s'}(a_{t+1})) \tag{3.9}$$

is often called the effective discount factor (EDF). The fact that the EDF is lower when the next period marginal propensity to consume ($c'(a_{t+1})$) is lower is easy to interpret. The intuition is, that when the MPC is high, self $t+1$ is going consume a large share of the extra saving which self t can make today. In this case, only the short term discount factor ($\beta\delta$) is important, as self t cannot really influence the future after $t+1$. When the MPC is low, self t also considers the impact of her decision after period $t+1$ too. Also, in the case of full naiveté ($\widehat{\beta} = 1$) self t uses the short term discount factor regardless of MPC, as the agent believes that future selves will discount exponentially, and thus behave in a way that is optimal to self t. Another interpretation is that she does not understand the time inconsistency problem, thus she only considers the tradeoff between the current period and next period thinking that self $t+1$ will make the optimal decision for both of them.

3.3 Characterisation of the equilibrium

As mentioned in the previous chapter, nothing guarantees that the consumption function is differentiable (or even continuous), so the Euler-equation may not hold. There are two ways to continue with the analysis. One is to analytically derive the necessary conditions, under which the consumption function is well-behaved, as in either Harris & Laibson (2001) or Harris & Laibson (2003). The other way is to consider the kind of pathologies that might arise, and examine whether the numerically derived consumption functions produce those pathologies, similarly to Harris & Laibson (2003). In this section I will briefly refer to the results of the former approach and provide intuition how those may apply to my model with naiveté. After that for my main analysis I will make sure to choose parameter values such that the consumption functions are well-behaved.

First of all the Bellman-equation of the naive hyperbolic consumer needs to be derived. The derivation is quite similar to the Euler-equation in the previous section. The current value and continuation value functions are defined in a similar way as before. From the perspective of self t:

$$W^s_{t+1}(a_{t+1}) = u(c^s_{t+1}(a_{t+1})) + \widehat{\beta}\delta \mathrm{E}_{t+1} V^{s'}_{t+1}(R(a_{t+1} - c^s_{t+1}(a_{t+1})) + e_{t+2}) \tag{3.10}$$

$$V^s_t(a_{t+1}) = u(c^s_{t+1}(a_{t+1})) + \delta \mathrm{E}_{t+1} V^{s'}_{t+1}(R(a_{t+1} - c^s_{t+1}(a_{t+1})) + e_{t+2}) \tag{3.11}$$

and analogously to equation 3.3 an equation linking W^s_{t+1} and V^s_t can be derived:

$$V^s_t(a_{t+1}) = \frac{1}{\widehat{\beta}} W^s_{t+1}(a_{t+1}) + \frac{1-\widehat{\beta}}{\widehat{\beta}} u(c^s_{t+1}(a_{t+1})). \tag{3.12}$$

The envelope condition for W^s_{t+1} is

$$u'(c^s_{t+1}(a_{t+1})) = (W^s_{t+1})'(a_{t+1}) \tag{3.13}$$

and W_t can be defined as

$$W_t^s(a_t) = \max_{c \in \Gamma_s(a_t)} u(c) + \beta\delta \mathrm{E}_t V_t^{s'}(R(a_t - c) + e_{t+1}). \tag{3.14}$$

Substituting in equation 3.12 it becomes

$$W_t^s(a_t) = \max_{c \in \Gamma_s(a_t)} u(c) + \frac{\widehat{\beta}\delta}{\beta} \mathrm{E}_t \left[(W_{t+1}^{s'} - (1 - \widehat{\beta}) u \circ c_{t+1}^{s'})(R(a_t - c) + y_{t+1}) \right] \tag{3.15}$$

and finally $c_{t+1}^{s'}$ can be substituted out using the envelope condition. The final Bellman equation is

$$W_t^s(a_t) = \max_{c \in \Gamma_s(a_t)} u(c) + \frac{\widehat{\beta}\delta}{\beta} \mathrm{E}_t \left[\left(W_{t+1}^{s'} - (1 - \widehat{\beta}) u \circ (u')^{-1} \circ (W_{t+1}^{s'})' \right) (R(a_t - c) + y_{t+1}) \right] \tag{3.16}$$

Again, the Bellman-equation derived in Harris & Laibson (2003) is a special case of it with $\widehat{\beta} = \beta$. More importantly, the only difference between this naive hyperbolic Bellman-equation and their sophisticated hyperbolic Bellman-equation is some constants, so their analysis also applies here.

For the purposes of this thesis the most important result is theorem 30. in Harris & Laibson (2001). Adopted to this model of naiveté, it states the following:

Theorem. Suppose that $R\widehat{\beta}\delta/\beta < 1$. Then there exists $\underline{\widehat{\beta}} \in [0, 1)$ and $\bar{A}_s \in (0, \infty)$ such that for all $\widehat{\beta} \in \left[\underline{\widehat{\beta}}, 1\right]$ and all equilibria c_s:

i) $R(a - c_s(a)) + e \in [e_l, \bar{A}_s]$ for all $a \in [e_l, \bar{A}_s]$ and $e \in E$

ii) c_s is Lipschitz-continuous on $[e_l, \bar{A}_s]$

The proof is omitted as it is very technical and there is no additional intuition to the sophisticated hyperbolic case in Harris & Laibson (2001). The main result here is that the continuity of the consumption function depends on the beliefs about the hyperbolic discount factor in the future instead of the actual hyperbolic discount factor. However as I will show in the appendix it is perfectly consistent with the intuition on why discontinuities may arise.

To get some intuition on why these irregularities emerge in the first place, consider a deterministic version of this model. Suppose that for some reason there is a kink in the consumption function of the agent at time t (for example because the liquidity constraint is binding for some part). It is shown in appendix A that self $t-1$ either wants self t to stay well below the kink, or he wants to push him over the kink. In the latter case he is willing to cut his own consumption to make self t choose a consumption over the kink, which produces a downward discontinuity. So the discontinuity is a result of intra-personal strategic interaction. This fact explains why the beliefs on the future hyperbolic discount factor matter and not the actual discount factor: actions are based on beliefs, and self $t-1$ believes that self t has $\beta = \widehat{\beta}$ when she makes the – game theoretic – consumption decision.

To assess how serious these pathologies are Harris & Laibson (2003) performed simulations for various parameter values. They find that (as proven formally) irregularities vanish as β gets close to 1. In the naive case the expectation is that $\widehat{\beta}$ needs to get close to one for this to happen. They further find that a higher risk-aversion coefficient in the utility function and a higher variance in income uncertainty also mitigates the pathologies described above. Most importantly, their result is that for empirically relevant parameter values the consumption function is usually well-behaved: monotone increasing, continuous, and concave.

Chapter 4

Results

4.1 Calibration

The calibration of this model is based on the calibration in Huggett (1993) (which is in turn based on Lucas (1980)). The main reason is to make the models easily comparable.

There are six periods per year. The low state is interpreted as a period of unemployment and the high state is interpreted as employment. $e_h \equiv 1$, while e_l is the unemployement benefit as a fraction of the employment wage. In the baseline calibration $e_l = 0.1$. The transition probabilities are choosen such that the model roughly matches the average unemployment duration and standard deviation of hours worked observed in US data: $\pi_{hh} = 0.925, \pi_{lh} = 0.5$. The borrowing limit is set to $\underline{a} = -2$, which corresponds to 4 months' wage. Later on I will try different values for e_l and $\underline{a}$ to see how these parameters affect equilibrium interest rates.

The risk-aversion coefficient of the CRRA utility function is set to $\gamma = 3$. It is on the high side of the usual estimates, but not implausibly high. For example Havránek (2015) performs meta-analysis of the existing literature, and finds that the elasticity of intertemporal substitution (EIS) is well below one. He deduces that micro estimates of the EIS for asset holder are around 1/3, which implies $\gamma = 3$ for the CRRA utility function. Also, most of these estimates assume a hyperbolic discounting model, so adhering to the conventional $\gamma \approx 1$ might not be strictly necessary. The main reason for the relatively high γ, however, is the previously mentioned fact, that a higher coefficient mitigates the discontinuity problems of the consumption function. With $\gamma = 3$ none of the parameter sets produce problems. Thus I propose that $\gamma = 3$ is a sensible value for this model until we have structural estimates for said parameter using models of hyperbolic discounting.

The hyperbolic discount factor is set to $\delta = 0.9932$. It implies an annual discount factor of around 0.96 which is sensible. The hyperbolic discount factor is set to $\beta \in [0.6, 1]$ and the coefficient of naiveté is $\widehat{\beta} \in [\beta, 1]$. These values are low enough to make a difference ($\beta = 0.6$ implies that agents value next period – which is only two months away – more than 40 % lower than today), but high enough for the consumption functions to be well behaved.

The model parameters are summarized in table 4.1.

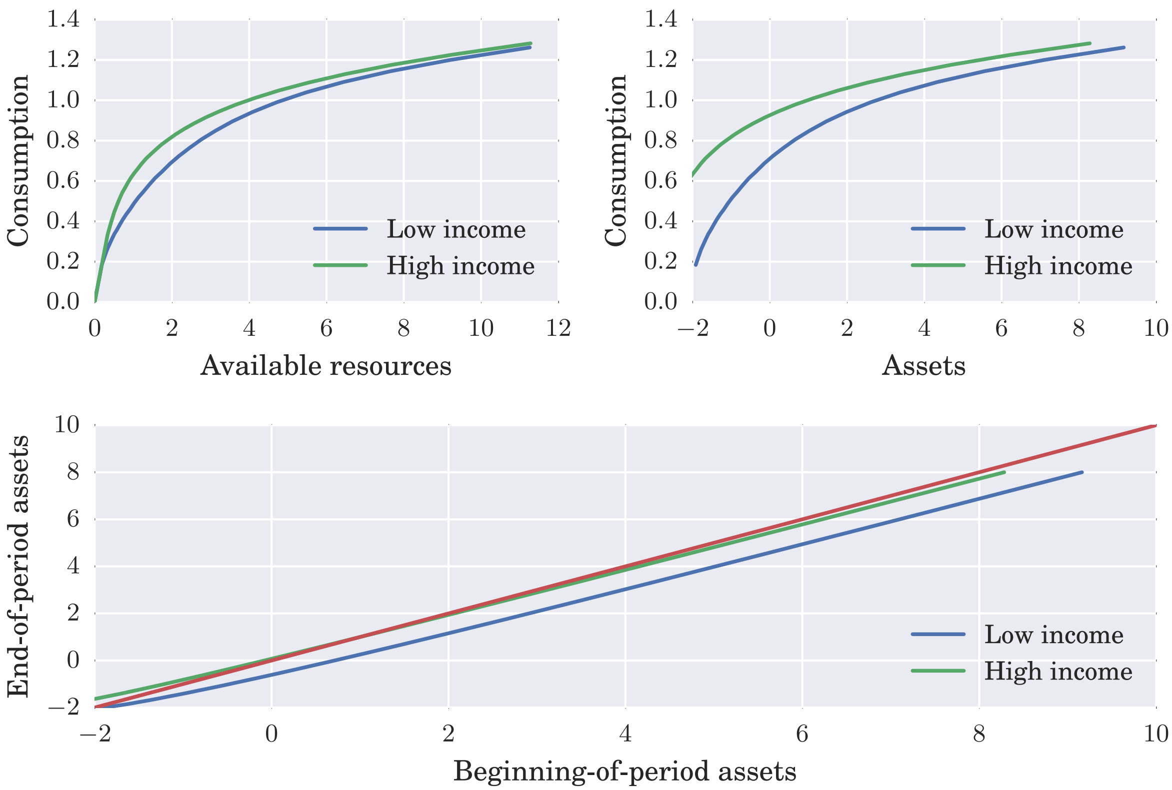
Table 4.1: Baseline calibration

Income	Preferences
$e_l = 0.1$	$\beta \in [0.6, 1]$
$e_h = 1$	$\widehat{\beta} \in [\beta, 1]$
$\pi_{lh} = 0.5$	$\delta = 0.9932$
$\pi_{hh} = 0.925$	$\gamma = 3$
$\underline{a} = -2$	

As described in the previous chapter, the consumption function may become discontinuous at low values of $\widehat{\beta}$. This problem does not arise with the above set of parameters. To show this, the policy functions of the most susceptible case ($\widehat{\beta} = 0.6$) are plotted on figure 4.1. They are monotone, continuous and concave at the equilibrium interest rates, which implies that all other policy functions should be fine too (as $\widehat{\beta} > 0.6$ in every other case).

Figure 4.1: Policy functions for $\beta = \widehat{\beta} = 0.6$

Equilibrium APR $= -9.38\%$



4.2 Solution method

Numerically calculating the equilibrium interest rates is relatively straightforward. The procedure is a direct translation of the definition of the equilibrium. It goes as follows:

1. Calculate the policy functions $a(x)$ given r using value-function iteration (with endogenous grid-points for better speed and accuracy).

2. Calculate the excess asset demand for the invariant distribution with given r and $a(x)$. It is achieved by simulating the income and consumption of N agents for T periods, then calculating $\sum_{i=1}^{N} a_{i,T}$. In the simulations for this thesis $N = 5000$, $T = 1000$.

3. Try another r and repeat steps 1 and 2 until convergence.

For step 1 a grid of 30 points on the interval $[\underline{a}, \underline{a}+10]$ is used. Values of the consumption function between grid points are calculated using Piecewise Cubic Hermite Interpolating Polynomials, which is a version of piecewise cubic splines with the important distinction that it preserves the monotonicity in the data. An advantage of using piecewise polynomials is that the first derivative of the consumption function is continuous and readily available at any point. It is important as the marginal propensity to consume is necessary for the calculation of the effective discount factor.

Step 2 is simple aside from a small trick: at each time period the law of large numbers is enforced, so the distribution at any point in time is guaranteed to be the stationary distribution of the Markov-process for income. This reduces the inaccuracy stemming from the fact that a finite number of agents is used. After getting the excess asset demand, the new value for r is chosen by a root-finding algorithm (Powell's hybrid method).

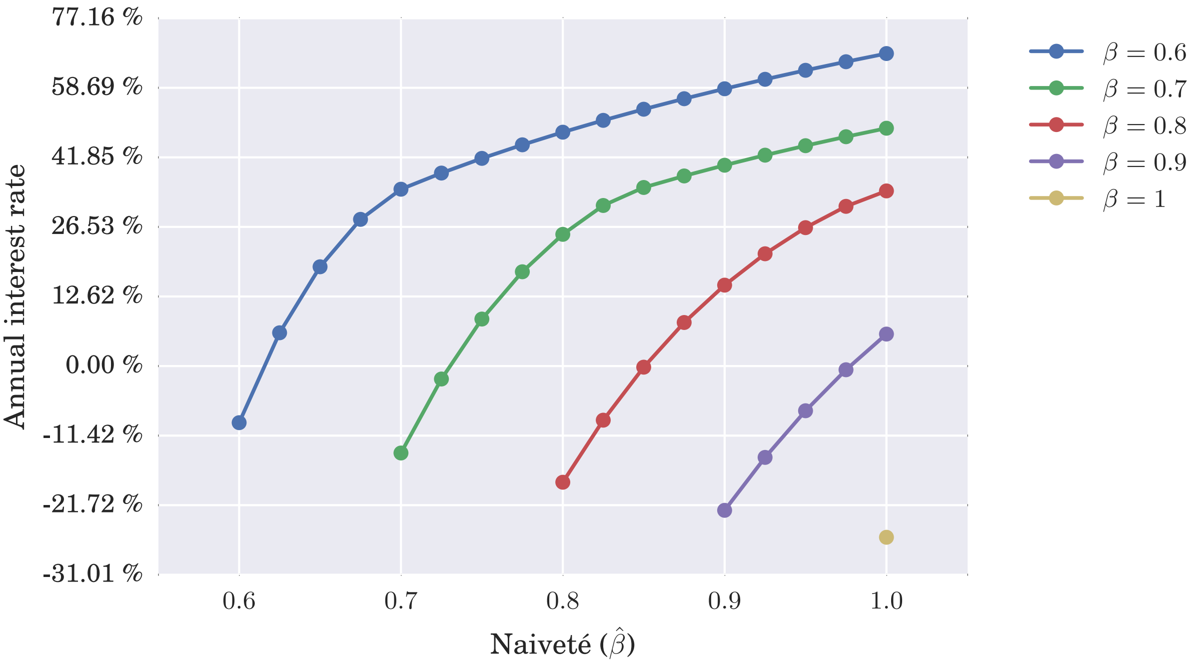
4.3 Equilibrium interest rates

4.3.1 Baseline calibration

Equilibrium interest rates are computed for 5 empirically relevant levels of β (0.6, 0.7, 0.8, 0.9, 1). Figure 4.2 plots the equilibrium interest rates against naiveté for each β.

The most obvious observation is that interest rates have a huge range starting from implausibly low values to extremely high ones (although the latter are not unusual in credit card lending). At $\beta = 1$ the model is the same as in Huggett's original paper, and the equilibrium interest rate calculated here roughly matches his equilibrium interest rate of -23%. To understand the negative interest rates note that the risk-free bond is the only asset that the agents can use to smooth their consumption. As there are no state-contingent assets, the bond also act as an insurance device with which holders can mitigate some of their idiosyncratic risk. Also remember that the risk-aversion parameter of the utility function is

Figure 4.2: Equilibrium interest rates – baseline calibration



relatively high. In fact so high that the baseline calibration with an exponential discounting model produces an implausibly low risk-free interest rate. At the same time, the baseline model can be consistent with the observed interest rates if people are at least partially naive hyperbolic discounters. It highlights the fact that econometric estimates are needed for the preference parameters $(\beta, \widehat{\beta}, \delta, \gamma)$ based on a model with hyperbolic discounting to allow for the reliable calibration of models such as this.

Both β and $\widehat{\beta}$ have a big influence on the interest rate, and the direction of their effects is as expected. β measures the value of future consumption relative to current consumption. If it is lower, agents want to consume more today, and have less savings. To make them save enough for a zero asset balance they need to be incentivized to do so, and for that a higher interest rate is needed. $\widehat{\beta}$ captures the beliefs about the patience of later selves. If the agent believes that in the future she is going to be more patient, then there is less need to save now – the future selves will do it for her. So again, she needs incentives to save more, thus the interest rate needs to be higher. Maybe somewhat surprisingly $\widehat{\beta}$ seems to be relatively more important, especially at the lower interest rates. It is especially remarkable taking into account the fact that agents differing only in $\widehat{\beta}$ have actually the same preferences, and as a consequence their optimal strategies would be the same. This result is in line with the growing body of literature showing that even minor levels of naiveté can have a profound effect on the equilibrium outcome (DellaVigna & Malmendier, 2004; Heidhues & Kőszegi, 2010).

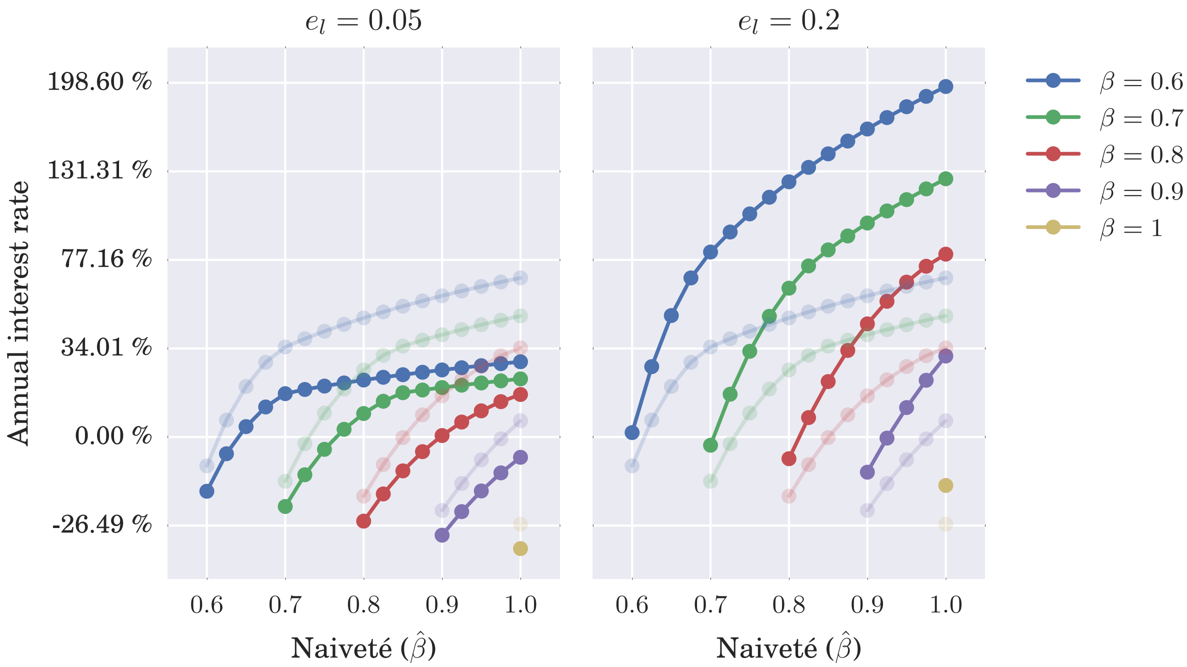
The fact that the increase in interest rates as naiveté increases gets relatively flat as interest rates get really high might be explained by the fact that the natural borrowing limit becomes effective at 5% per period (34% APR). It suggests that the natural borrowing limit can also be an important channel through which equilibrium interest rate is determined.

4.3.2 Alternative unemployment benefits

The income in the low state is interpreted as unemployment benefits compared to the wage earned in periods of employment. Knowing how changing it affects interest rates is both an interesting theoretical and an important practical question. Figure 4.3 shows the equilibrium interest for two alternative values for low state income: 0.05 and 0.2.

The results show that unemployment benefits greatly influence interest rates – especially for naive populations. One obvious mechanism through which they do so is changing the natural borrowing limit. For $e_l = 0.05$ the NBL becomes effective at 15% APR, which is a relatively low value. It explains why there seems to be an upper limit to interest rates, after which additional naiveté or impatience does not really matter: the NBL is a strong constraint regardless of preferences. It also explains the fact that higher interest rates more than double in case of an increased e_l: the NBL was a strong constraint in the baseline model, but in this case it is not. The other factor through which unemployment benefits influence interest rates is that when they are relatively high, there is less need for consumption smoothing

Figure 4.3: Equilibrium interest rates – alternative unemployment benefits

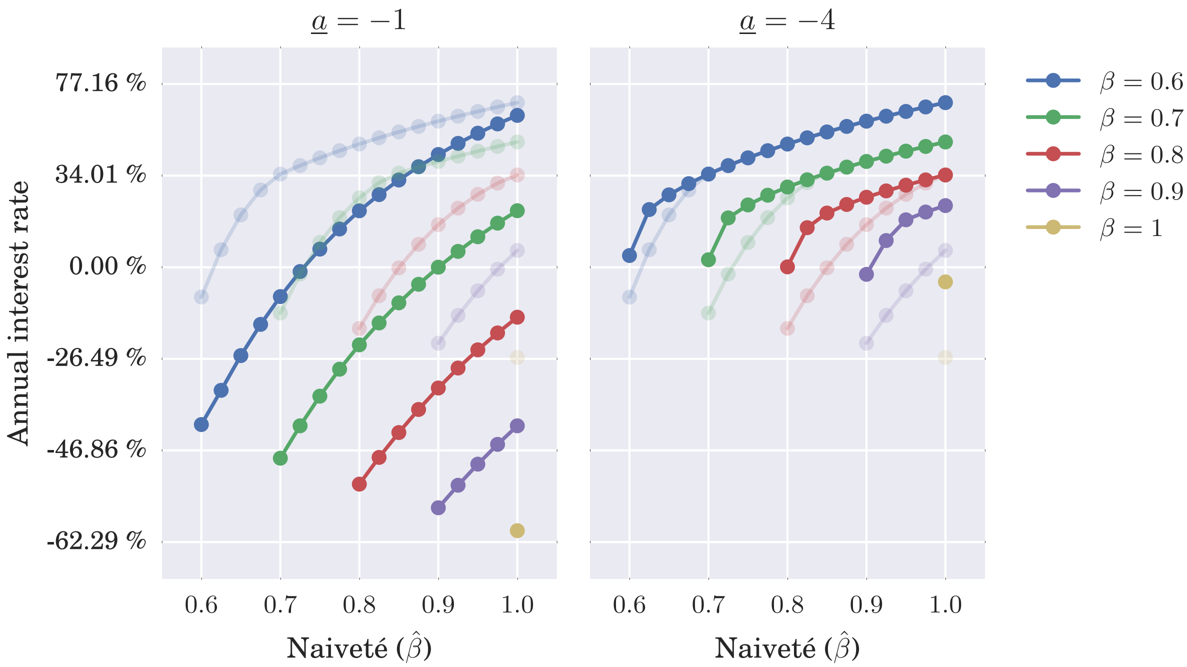


and insurance, therefore less demand for the asset. That of course decreases the price of the bond, or in other words increases its interest.

4.3.3 Alternative borrowing constraints

Making the borrowing limit stricter has an unexpectedly stark effect – equilibrium interest rates in this case are extremely low. The interest rate decreases, because (1) agents cannot accumulate large amounts of debt (2) agents are more likely to run into the borrowing constraint in later periods, thus they want to keep more savings. On the other hand easing the borrowing constraint does not have such an extreme effect, but it is not negligible either. At interest rates above 34% the less stringent borrowing constraint has absolutely no effect, as the NBL is above -2. For lower interest rates it has a relatively big effect, but not as big as the difference for exponential discounters would suggest.

Figure 4.4: Equilibrium interest rates – alternative borrowing constraints



4.4 Asset distribution

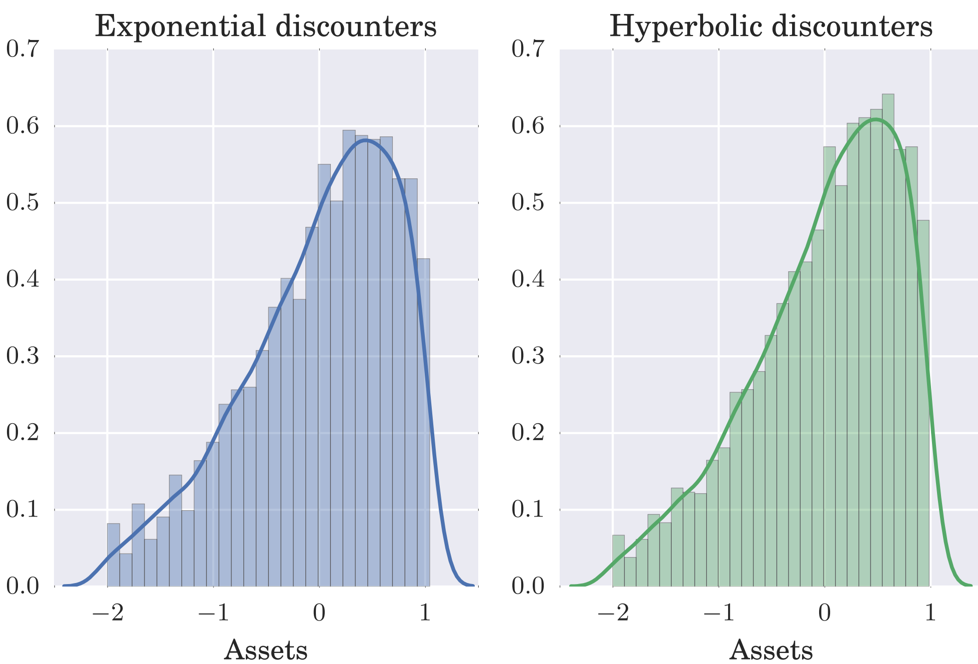
This section deals with asset distributions in a heterogeneous (in terms of discounting parameters) populations. The heterogeneity can either be in the hyperbolic discount factor (β) or in the level of naiveté ($\widehat{\beta}$). The huge differences in interest rates that these parameters can cause suggest that the differences in distributions might also be large, but it will be instructive to see them directly nevertheless.

4.4.1 Heterogeneity in impatience

In this specification 20% of the population is sophisticated hyperbolic discounter with $\beta = 0.7$ and the other 80% is exponential discounter. The value for β is chosen because it is often considered a plausible value in the behavioral literature. Also there is empirical evidence supporting this: Laibson *et al.* (2007) estimate $\beta = 0.703$ using MSM. The choice of $20\% - 80\%$ on the other hand is quite arbitrary. The motivation for a smaller share of hyperbolics is to conform to standard models with exponential discounting, and to illustrate what happens to a hyperbolic minority in that case.

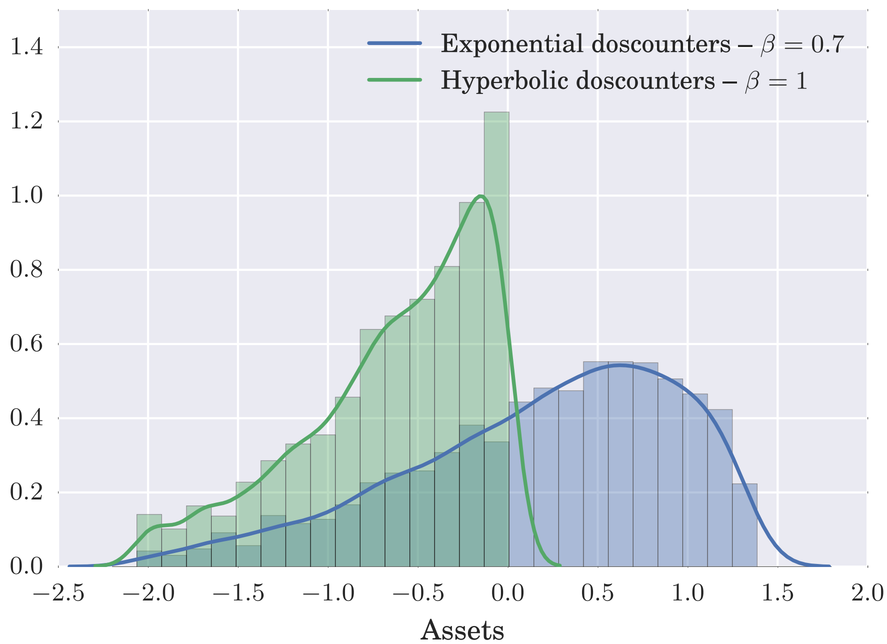
As a benchmark the distributions of the homogeneous populations are displayed on figure 4.5. The two distributions are essentially the same. This might seem strange at first, but taking into account the fact that it is a general equilibrium model, it is not that surprising. The interest rates must adapt such that the excess asset demand is zero in both cases It is why the support of the two distributions is almost the same. The similar shape can be explained by the fact that the difference between the behavior of exponential discounters and sophisticated hyperbolic discounters is not enormous. Remember, in case of $c'(a) = 0$ their Euler-equations would be exactly the same.

Figure 4.5: Asset distributions – benchmark for heterogeneity in β



In the heterogeneous population the interest rate is such that the combined asset balance of this population is zero. The resulting annual interest rate in this case is −23.8%, which is much closer to the −26.1% of exponentials than to the −14.1% observed for hyperbolics. It is not surprising given that the population is dominantly made up of exponential discounters. It is also not that far from either, which explains the fact that there is no dramatic difference between their respective asset distributions as shown on figure 4.6.

Figure 4.6: Asset distributions – heterogeneity in β



Also note that the hyperbolic discounters are sophisticated in this case, so they are choosing the strategy that maximizes their welfare with the restriction that it is subgame-perfect. It is a second best option, as the first best strategy would need some form of commitment. However, even if self t had access to a commitment device, then self $t+1$ would be worse off, as he prefers his current strategy to the one self t would commit to. Therefore even though there is a difference in wealth distribution between the two groups, it does not necessitate policy intervention as (1) it reflects differences in preferences (2) simple interventions, such as offering a commitment device, would not increase the welfare of *every self* of the agent.

Another intuition for the fact that hyperbolics may not be worse off in the mixed population than alone is that they are relatively impatient, while exponential discounters are much

Table 4.2: Mean expected utilities – heterogeneity in β

	Exponentials	Hyperbolics
Homogeneous populations	-134.16	-98.09
Heterogeneous population	-129.56	-116.81

more patient. It is shown by the fact that in their homogeneous populations the interest rate of the hyperbolics is significantly higher than that of the exponentials. When they are paired, hyperbolic discounters provide an investment opportunity for the excess money of exponential discounters, while hyperbolic discounters gain a source of credit from which they can cover their impatient consumption. The two groups seem to complement each other.

To test this intuition I calculated the mean expected utilities based on the simulation results. They are summarized in table 4.2. Note that the utilities cannot be compared across groups as the utility functions are different. However, they can be compared across models. Contrary to the apparent complementarity between the two groups, the result is that hyperbolics would prefer to be alone, while exponentials are better off in the mixed population. A possible explanation is the following: interest rates in the homogeneous hyperbolic case impose some degree of self-control, and the hyperbolics actually act almost like exponentials (remember the similar asset distributions). In the mixed population the interest rates are lower, and at these low interest rates hyperbolic discounters do not have any savings, they opt for instant gratification.

To assess inequality, the Gini coefficient of the populations are displayed in table 4.3. The indices are relatively low for homogeneous populations. In the combined case the intra-group inequalities remain essentially the same. The Gini index of the whole population increases slightly – as in this case there are two groups choosing different strategies and thus having different asset distributions – but still remains moderate.

Table 4.3: Gini coefficients – heterogeneity in β

	Exponentials	Hyperbolics
Homogeneous populations	0.2	0.19
Heterogeneous population	0.19	0.21
— combined	0.24	

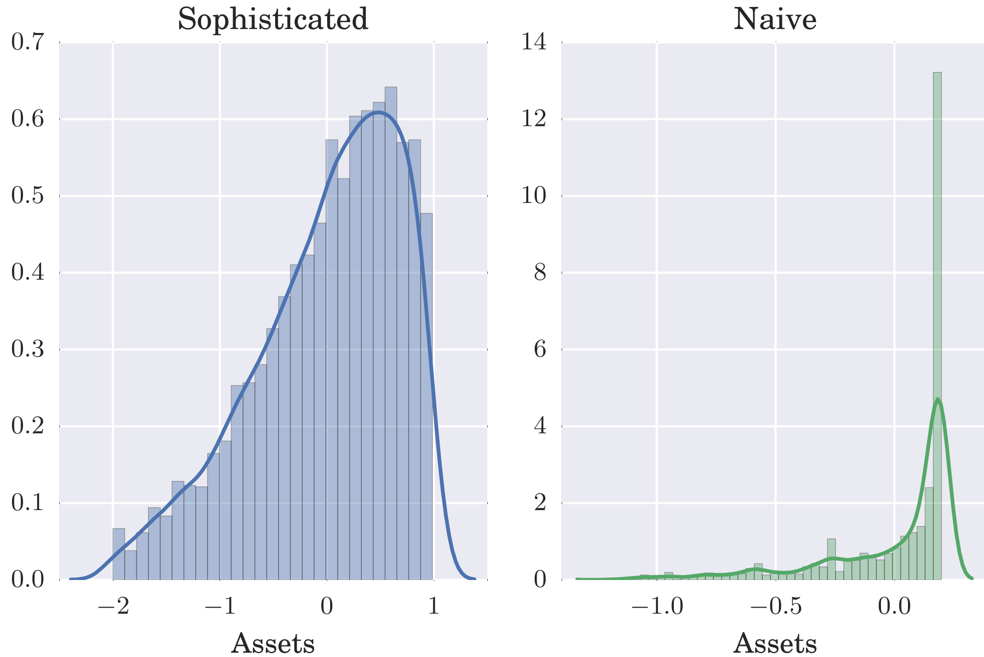
4.4.2 Heterogeneity in naiveté

In this population the difference between the agents is the level of naiveté. There are two groups. Both are hyperbolic discounters with $\beta = 0.7$, but the larger group (80%) consists of sophisticates, while members of the smaller group (20%) are naive. This scenario is much more interesting, as the groups actually have the same preferences. The only difference is their information about it. It also implies that the welfare-maximizing strategy – and thus the resulting asset distributions – for these two groups should be the same: the strategy chosen by the sophisticates.

The asset distributions of homogeneous populations are shown on figure 4.7. In this case the benchmark distributions are already strikingly different. The naifs' distribution starts above -2 as the interest rate in that case is so high that the natural borrowing limit is effective. Also, the maximum of the distribution is around 0.5, showing that even with huge interest rates naifs do not wish to make savings – they think they do not have to because they will do it in the future.

The annual equilibrium interest rate in the mixed population is -3.7%, which is quite far from the equilibrium interest rates for pure populations with these parameters (-14.1% and 48.7%). As it suggests, the difference in the asset distributions is huge (figure 4.8). This difference is especially stark in light of the fact that these two groups actually have the same preferences and the same optimum. Also notice that all of the naifs are borrowers while most of the sophisticates are lenders. With negative interest rates it means that sophisticates are paying interest rates to the naifs in each period, essentially subsidizing them. Although at

Figure 4.7: Asset distributions – benchmark for heterogeneity in $\widehat{\beta}$



3.7% this subsidy is most likely negligible.

It is interesting to contrast this specification of the model with the previous one (heterogeneity in naiveté). Pairing the two groups looked symbiotic (although it turned out it is not), as their preferences were complements of each other. In this case it is not the case. To quantify the welfare effects, mean expected utilities are again shown in table 4.4. Now the welfares are also comparable across groups as they have the same preferences. As expected, naifs have a lower mean utility even when they are alone, as they make suboptimal saving decisions. However, when the two groups are paired, the difference becomes even more striking. The welfare of sophisticates increases at the cost of the welfare of naifs. Its mechanism is simple: interest rates decrease for the naifs making them save even less and the opposite happens for sophisticates. This leads to the naifs making even worse decisions than before and the sophisticates saving more due to increased interest rates.

The resulting large inequalities are also important. As shown in table 4.5 income inequal-

Figure 4.8: Asset distributions – heterogeneity in $\widehat{\beta}$

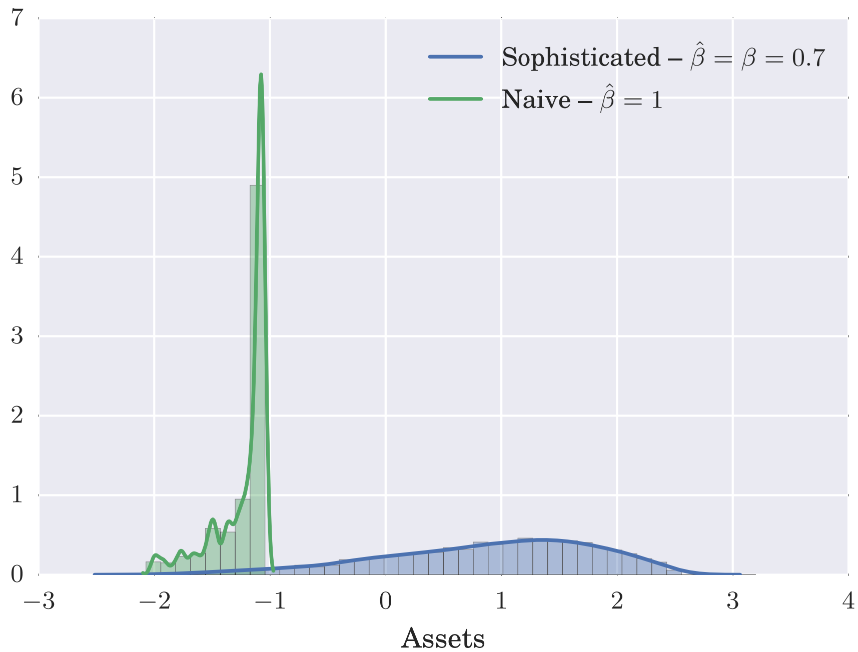


Table 4.4: Mean expected utilities – heterogeneity in $\widehat{\beta}$

	Sophisticates	Naives
Homogeneous populations	-133.19	-98.09
Heterogeneous population	-187.30	-82.26

ities are quite small in the separate populations. It is especially true for the naive population as the support of its asset distribution (the ergodic set) is narrow. In the heterogeneous case intra-group inequalities are still low, but the Gini-coefficient of the combined population increases significantly. Actually the simulated Gini coefficient in this case is comparable to

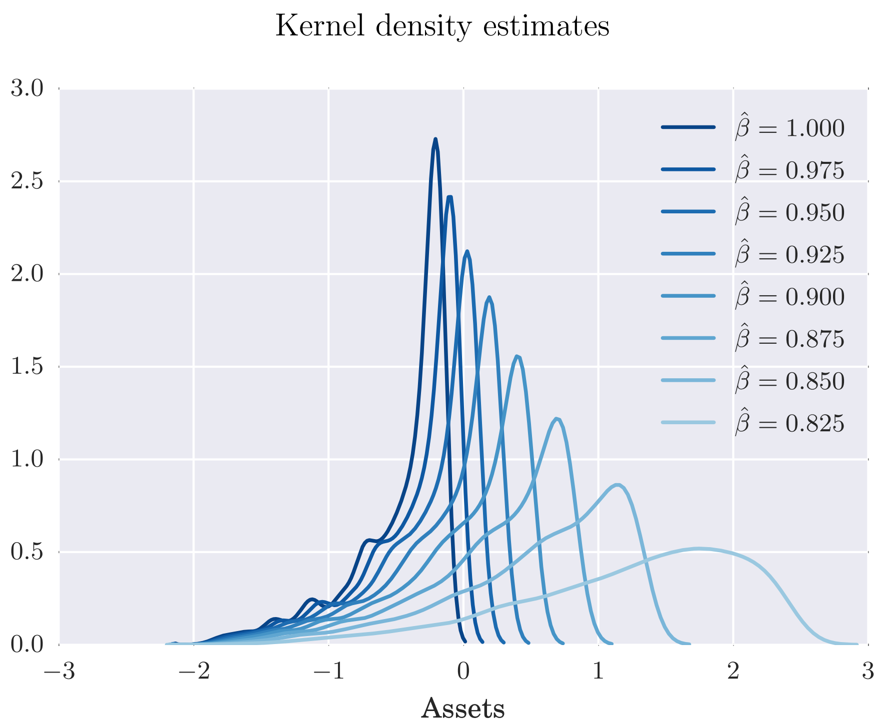
Table 4.5: Gini coefficients – heterogeneity in $\widehat{\beta}$

	Sophisticates	Naifs
Homogeneous populations	0.19	0.07
Heterogeneous population	0.18	0.14
— combined	0.38	

the estimated Gini index of the US (US Gini coef. in 2013: 41.4; World Bank data). It is not to say that it is an accurate model of income inequalities, the shape of the distribution is not even similar to what is observed in data – but it showcases that differences in naiveté can explain high levels of inequality even with everything else being equal.

To demonstrate this phenomenon with a different example, consider a population with $\beta = 0.8$ and naiveté distributed uniformly between β and 1 ($\widehat{\beta} \sim \mathrm{U}[\beta, 1]$). In practice due to computational constraint it is represented as a discrete uniform distribution with 8 elements. Also $\widehat{\beta} = 0.8$ is omitted, as there is no interest rate at which the asset balance is zero and the fully sophisticated agents have finite expected utility with their equilibrium strategy. Figure 4.9 displays the wealth distributions and table 4.6 displays the mean utilities at the equilibrium interest rate (15%).

Figure 4.9: Asset distributions – uniformly distributed $\widehat{\beta}$



Despite the fact that the hyperbolic discount factor is relatively close to one – therefore there cannot be large differences in naiveté – the differences in asset distribution are sig-

Table 4.6: Mean expected utilities – uniformly distributed $\widehat{\beta}$

$\widehat{\beta}$	0.825	0.850	0.875	0.900	0.925	0.950	0.975	1.000
$E[\mathcal{U}]$	-89.35	-98.23	-105.51	-111.98	-117.91	-123.42	-128.59	-133.46

nificant. Again, naifs are on the lower end of the asset distribution, while sophisticates are at the top. In this case the risk-free rate is positive and high, so income is continuously redistributed from naifs to sophisticates via interests. This result echoes the conclusions of Heidhues & Kőszegi (2010) in a different context: even small differences in naiveté can have severe welfare implications.

This result also has serious policy implications, as wealth redistribution from people with lower welfare to people with higher welfare is generally not favorable. However, proposing tools with which this problem could be addressed is not an easy task. Consider for a moment the – probably infeasible – scheme of educating naifs to become sophisticated. In a population with no heterogeneity in naiveté, this would obviously be a libertarian paternalistic policy, which would also improve everyone's welfare. On the other hand in a mixed population the sophisticates benefit from the naiveté of the naifs – there is no way to help the latter group without hurting the former.

Introducing a commitment device could also have an unanticipated outcome. Sophisticates would use it (increasing their welfare at the time of introduction, and slightly decreasing it afterward), but naifs would not, as they do not recognize the time inconsistency of their preferences. This would mean that gap in asset distributions would widen, and as a consequence there would be even more income redistribution. This is surprising, as providing a commitment device for hyperbolic discounters is usually associated with welfare gains (Laibson, 1998).

Chapter 5

Conclusions

5.1 Future research

5.1.1 Robust numeric methods for discontinuous consumption functions

Although as shown in section 4.1 the consumption function is well-behaved for reasonable parameter values, having robust methods to numerically approximate the value and consumption functions would open the way for examining a number of situations of interest. There are variations of this model in which discontinuity is not caused by extreme parameter values, but by some other feature of the model. Remember that the discontinuity at time t was caused by a kink in the continuation value function at time $t+1$. It is not difficult to come up with settings in which it happens.

Using exogenous grid points and iterating on the value function instead of the Euler-equation might be helpful, as the value function should not be discontinuous (even if it has kinks). However, this induces a serious performance penalty, as it necessitates performing

n non-linear maximizations or root-finding operations per iteration, where n is the number of grid points. Harris & Laibson (2001) show that even in the case of pathologies, the consumption function should only be discontinuous at a finite number of points. Hopefully this property can be leveraged to create an algorithm that handles discontinuities while not abandoning endogenous grid points.

5.1.2 Wedge between borrowing and lending rates

An interesting question is what happens in the economy if borrowing rates are fixed at a relatively low level. How high do lending rates need to be to have zero net asset demand? While this question is easy to answer in similar models with exponential discounting, unfortunately the consumption function is not well-behaved in the hyperbolic case.

The main problem is that the consumption function is very likely to be discontinuous. It is easy to intuitively understand the reason. It is a standard result of intertemporal consumption-saving models with different borrowing and lending rates, that self t will have a kink in her consumption function, even without hyperbolic discounting. As shown in appendix A, a kink in the consumption function at time t implies a discontinuous consumption function at time $t-1$ if $\widehat{\beta} < 1$. Therefore developing methods to deal with discontinuous consumption functions would also be a huge help in incorporating a wedge between borrowing and lending rates.

5.1.3 Multiple assets

Similar models could also be used to price assets other than the simple non-contingent risk-free bond. In hyperbolic discounting models, the study of illiquid assets is of particular interest, because they can act as a commitment device to restrict the choices of future selves. Laibson (1997) calls these assets golden eggs because of this property.

In this framework it is relatively simple to model having an illiquid asset instead of the one-period risk-free bond, if illiquidity is defined as an adjustment cost. Seeing how the equilibrium interest rates change in response to the level of illiquidity would provide valuable information. However, a much more relevant question would be the following: what the liquidity premium is when *both* liquid and illiquid saving devices are available.

There have been such studies of life cycle savings (Angeletos *et al.*, 2000), but interest rates are usually exogenous in these models. To my knowledge as of yet there is no published general equilibrium model with both liquid and illiquid assets. Theoretically, solving such a model in the framework described in this thesis should be possible using value function iteration (with exogenous grid points), but there are a few caveats one has to be aware of. To show these consider the outline of an extended model below.

The Bellman-equation is (analogously to equation 3.16)

$$
\begin{aligned}
W^s(a,z) = & \max_{(\tilde{a},\tilde{z})\in\Gamma_s(a,z)} u(h(\tilde{a},\tilde{z})) \\
& + \frac{\widehat{\beta}\delta}{\beta}\mathrm{E}\left[(W^{s'} - (1-\widehat{\beta})u\circ g^{s'})(R_a(a+\tilde{a}) + y_{t+1}, R_z(z+\tilde{z}))\right]
\end{aligned}
$$

where z denotes illiquid assets, g is the policy function and h is consumption, given that change in assets is $(\tilde{a},\tilde{z})$. Now the step to substitute out the policy function using the envelope condition requires that $(u\circ h)'$ has a left-inverse. This condition is satisfied, for example, if changing the illiquid asset incurs a convex adjustment cost. If it has a left inverse, then the Bellman-equation can be written as

$$
\begin{aligned}
W^s(a,z) = & \max_{(\tilde{a},\tilde{z})\in\Gamma_s(a,z)} u(h(\tilde{a},\tilde{z})) \\
& + \frac{\widehat{\beta}\delta}{\beta}\mathrm{E}\left[(W^{s'} - (1-\widehat{\beta})u\circ((u\circ h)')^{-1}\circ(W^s)')(R_a(a+\tilde{a}) + y_{t+1}, R_z(z+\tilde{z}))\right]
\end{aligned}
$$

and it is numerically solvable using value function iteration. However, it would be performance intensive to solve this at every step of a 2-dimensional non-linear root finding. Also, a much larger number of simulated agents would be needed to approximate a 2-dimensional asset distribution than the 5,000 used in this thesis.

5.1.4 Welfare analysis in mixed population models

The last sections of chapter 4 dealt with welfare and wealth distribution. It proved that these issues are significant and showed some results in a stylized way. It would be relevant from a policy standpoint to provide a rigorous discussion of these issues and to simulate the welfare effects of some hypothetical policy interventions and redistribution schemes. Also finding out welfare maximizing interest rates would be an interesting exercise (remember that in section 4.4.1 hyperbolic discounters were worse off paired with exponentials at the new equilibrium rates despite the intuition).

There is research on this topic (for example Krusell *et al.* (2002) deal with the welfare implications of hyperbolic discounting in an equilibrium model), and there are also papers on the welfare effects of naiveté in smaller-scale settings (for example Heidhues & Kőszegi, 2010), but there is very little research on the welfare effects of naiveté in general equilibrium models. Despite the difficulties of welfare analysis of time-inconsistent agents (which selves welfare do we consider?) it would be an important area of research, and the model used in this thesis is quite suitable for further investigation of this issue.

5.2 Summary

In this thesis I presented a general-equilibrium model with a highly stylized income side to analyze the interest rates in an economy populated by partially naive hyperbolic discounters. I showed that the hyperbolic discount factor, and more importantly, the level of naiveté has substantial effect on the equilibrium interest rates. Their influence depends on some of the model parameters (borrowing limit and unemployment benefits), but generally any interest rate (from well into the negatives to almost triple-digit numbers) can be an equilibrium with suitable discount factors. It also showed that while heterogeneous agent models *can* generate low risk-free interest rates, they do not necessarily do so under plausible levels of naiveté.

The thesis also dealt with the implications of naiveté for asset distributions in heterogeneous populations. The main result is that in a heterogeneous population the welfare of the agents can be much different than in their respective homogeneous populations. It is particularly true for heterogeneity in naiveté. Naifs – who have a lower welfare than sophisticates due to suboptimal decisions to begin with – are much worse off in combined populations. I also argued that it is difficult to design simple policies that address this problem. Estimating the distribution of naiveté and doing welfare analysis based on it could be instrumental in designing an approach to deal with this phenomenon.

Appendix

The source code of the simulation, figures and tables can be obtained at the Bitbucket site of the project: `https://bitbucket.org/stanmart/thesis/src`.

A Irregularities of the consumption function

I will illustrate that the consumption function might not be continuous and monotone increasing with the example from Harris & Laibson (2003). It also provides an argument for why $\widehat{\beta}$ causes the discontinuity and not the actual β. I will solve the consumption problem from the point of view of self $T-1$, so her beliefs apply.

Consider a deterministic, finite-horizon version of this model with $\underline{a} = 0$. The finite model can always be solved by backwards induction. If there is no remainder value then obviously $c_T = a_T$. Self $T-1$ knows this, so – according to self $T-1$'s beliefs – she will solve

$$\max_{c_{T-1}} u(c_{T-1}) + \widehat{\beta}\delta u(c_T)$$

$$\text{s.t. } a_T = R(a_{T-1} - c_{T-1}) + e_T \qquad \text{(Dynamic budget constraint)}$$

$$c_{T-1} \leq a_{T-1} \qquad \text{(Liquidity constraint)}$$

$$c_T = a_T \qquad \text{(Percieved strategy of self } T\text{)}$$

When the liquidity constraint does not bind, the first order condition for the above problem is

$$u'(c_{T-1}) = \widehat{\beta}\delta R u'(R(a_{T-1} - c_{T-1}) + e_T)$$

When the liquidity constraint binds, $c_{T-1} = a_{T-1}$. Let us denote the consumption function arising from this strategy by $\widehat{c}_{T-1}(a_{T-1})$. Now, self $T-2$ solves the problem

$$\max_{c_{T-2}} u(c_{T-2}) + \beta\delta u(c_{T-1}) + \beta\delta^2 u(c_T)$$

$$\text{s.t. } a_{T-1} = R(a_{T-2} - c_{T-2}) + e_{T-1} \qquad \text{(Dynamic budget constraint)}$$

$$c_{T-2} \leq a_{T-2} \qquad \text{(Liquidity constraint)}$$

$$c_{T-1} = \widehat{c}_{T-1}(a_{T-1}) \qquad \text{(Perceived strategy of self } T-1\text{)}$$

$$c_T = a_T \qquad \text{(Perceived strategy of self } T\text{)}$$

Now consider the continuation value function of self $T-2$:

$$V_{T-2}(a_{T-1}) = u(c_{T-1}(a_{T-1})) + \delta u(R(a_{T-1} - c_{T-1}(a_{T-1})) + e_T)$$

Also let $\widehat{a}$ denote the amount of assets below which in period $T-1$ the liquidity constraint is binding, and above which some marginal wealth is passed on to self T. It implies that in the region next to $\widehat{a}$

$$\lim_{a \to \widehat{a}-} V'_{T-2}(a) = u'(c_{T-1}(a))$$

$$\lim_{a \to \widehat{a}+} V'_{T-2}(a) = c'_{T-1}(a)u'(c_{T-1}(a)) + \delta R(1 - c'_{T-1}(a))u(c_T(a))$$

Finally at $a_{T-1} = \widehat{a}$ self $T-1$ (according to self $T-2$'s beliefs) is indifferent between marginal consumption in periods $T-1$ and T, that is

$$u'(c_{T-1}(\widehat{a})) = R\widehat{\beta}\delta u'(c_T(\widehat{a})).$$

By substituting it into the right limit of the continuation value of self $T-2$, we get that there is an upward kink in that function:

$$\lim_{a\to\widehat{a}-} V'_{T-2}(a) = u'(c_{T-1}(a)) < \left[c'_{T-1}(a) + \frac{1}{\beta}(1-c'_{T-1}(a))\right] u'(c_{T-1}(a)) = \lim_{a\to\widehat{a}+} V'_{T-2}(a)$$

The last thing to see is that setting a c_{T-2} such that $R(a_{T-2} - c(T-2) + e_{T-1} = a_{T-1} = \widehat{a}$ is never optimal. If

$$u'(c_{T-2}) < \beta\delta R \lim_{a\to\widehat{a}+} V'_{T-2}(a)$$

then self $T-2$ can increase her welfare by cutting consumption with marginal cost $u'(c_{T-2})$ and increasing savings with marginal benefit $\beta\delta R \lim_{a\to\widehat{a}+} V'_{T-1}(a)$. If on the other hand

$$u'(c_{T-2}) \geq \beta\delta R \lim_{a\to\widehat{a}+} V'_{T-2}(a)$$

then she can increase welfare by increasing consumption for a marginal benefit $u'(c_{T-2})$ and decreasing savings with a marginal cost $\beta\delta R \lim_{a\to\widehat{a}-} V'_{T-2}(a) < \beta\delta R \lim_{a\to\widehat{a}-} V'_{T-1}(a) \leq u'(c_{T-2})$. Therefore there is a set of a's such that self $T-2$ will never chooses them for a_{T-2}. She can only achieve this with a downward discontinuous consumption function.

Figure A.1 illustrates this example with $\beta = 0.7$, $\widehat{\beta} = 0.85$, $\delta = 0.99$, CRRA utility function with $\gamma = 1$ and $e = 1$.

B Tables of equilibrium interest rates

Figure A.1: Discontinuity of the consumption function

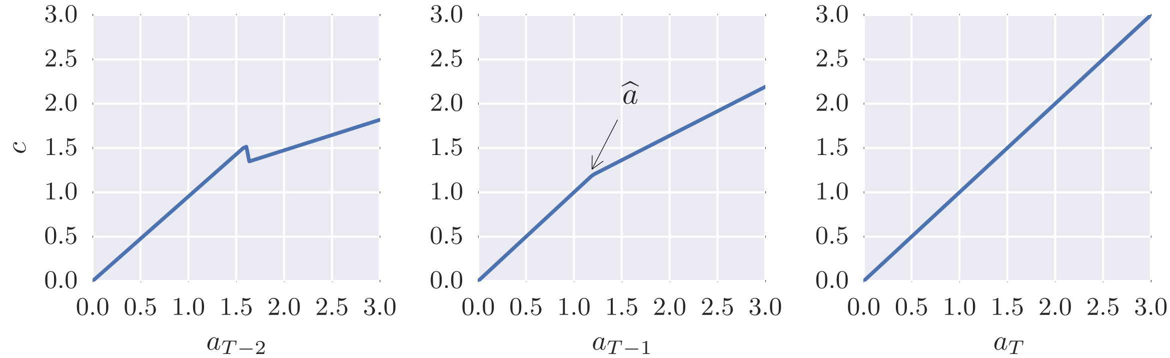


Table B.1: Equilibrium interest rates – baseline calibration

β / $\widehat{\beta}$	0.6	0.7	0.8	0.9	1.0
0.600	-9.38%	–	–	–	–
0.625	5.88%	–	–	–	–
0.650	18.39%	–	–	–	–
0.675	28.13%	–	–	–	–
0.700	34.66%	-14.09%	–	–	–
0.725	38.26%	-2.21%	–	–	–
0.750	41.62%	8.39%	–	–	–
0.775	44.77%	17.41%	–	–	–
0.800	47.76%	24.98%	-18.44%	–	–
0.825	50.61%	31.10%	-8.98%	–	–
0.850	53.33%	35.04%	-0.19%	–	–
0.875	55.94%	37.61%	7.75%	–	–
0.900	58.44%	40.05%	14.80%	-22.45%	–
0.925	60.85%	42.36%	21.00%	-14.76%	–
0.950	63.17%	44.57%	26.38%	-7.47%	–
0.975	65.41%	46.68%	30.91%	-0.64%	–
1.000	67.57%	48.71%	34.28%	5.65%	-26.13%

Table B.2: Equilibrium interest rates – $e_l = 0.05$

β / $\widehat{\beta}$	0.6	0.7	0.8	0.9	1.0
0.600	-17.03%	–	–	–	–
0.625	-5.54%	–	–	–	–
0.650	3.58%	–	–	–	–
0.675	10.62%	–	–	–	–
0.700	15.62%	-21.35%	–	–	–
0.725	17.23%	-12.12%	–	–	–
0.750	18.55%	-4.10%	–	–	–
0.775	19.77%	2.63%	–	–	–
0.800	20.93%	8.23%	-25.32%	–	–
0.825	22.02%	12.74%	-17.80%	–	–
0.850	23.05%	15.98%	-10.94%	–	–
0.875	24.04%	17.01%	-4.85%	–	–
0.900	24.98%	17.98%	0.50%	-28.97%	–
0.925	25.88%	18.89%	5.18%	-22.76%	–
0.950	26.74%	19.76%	9.20%	-16.94%	–
0.975	27.57%	20.59%	12.59%	-11.55%	–
1.000	28.36%	21.38%	15.29%	-6.68%	-32.33%

Table B.3: Equilibrium interest rates – $e_l = 0.2$

β / $\widehat{\beta}$	0.6	0.7	0.8	0.9	1.0
0.600	1.50%	–	–	–	–
0.625	26.34%	–	–	–	–
0.650	48.89%	–	–	–	–
0.675	67.55%	–	–	–	–
0.700	81.50%	-2.76%	–	–	–
0.725	92.93%	15.40%	–	–	–
0.750	103.79%	32.79%	–	–	–
0.775	114.21%	48.52%	–	–	–
0.800	124.28%	62.28%	-7.12%	–	–
0.825	134.03%	73.91%	6.77%	–	–
0.850	143.51%	82.70%	20.32%	–	–
0.875	152.73%	90.64%	33.16%	–	–
0.900	161.71%	98.25%	45.00%	-11.34%	–
0.925	170.47%	105.60%	55.75%	-0.38%	–
0.950	179.01%	112.71%	65.36%	10.40%	–
0.975	187.35%	119.61%	73.74%	20.83%	–
1.000	195.48%	126.32%	80.31%	30.78%	-15.33%

Table B.4: Equilibrium interest rates – $\underline{a} = -1$

β / $\widehat{\beta}$	0.6	0.7	0.8	0.9	1.0
0.600	-41.65%	–	–	–	–
0.625	-34.12%	–	–	–	–
0.650	-25.66%	–	–	–	–
0.675	-17.31%	–	–	–	–
0.700	-9.23%	-48.34%	–	–	–
0.725	-1.39%	-41.91%	–	–	–
0.750	6.09%	-35.46%	–	–	–
0.775	13.20%	-29.07%	–	–	–
0.800	19.96%	-22.87%	-53.04%	–	–
0.825	26.27%	-16.91%	-48.19%	–	–
0.850	32.19%	-11.10%	-43.30%	–	–
0.875	37.79%	-5.46%	-38.42%	–	–
0.900	43.08%	0.02%	-33.58%	-57.02%	–
0.925	48.07%	5.31%	-28.79%	-53.24%	–
0.950	52.75%	10.41%	-24.18%	-49.45%	–
0.975	57.13%	15.33%	-19.68%	-45.67%	–
1.000	61.24%	20.00%	-15.29%	-41.91%	-60.60%

Table B.5: Equilibrium interest rates – $\underline{a} = -4$

β / $\widehat{\beta}$	0.6	0.7	0.8	0.9	1.0
0.600	3.89%	–	–	–	–
0.625	20.45%	–	–	–	–
0.650	26.18%	–	–	–	–
0.675	30.70%	–	–	–	–
0.700	34.66%	2.47%	–	–	–
0.725	38.26%	17.29%	–	–	–
0.750	41.62%	22.22%	–	–	–
0.775	44.77%	26.03%	–	–	–
0.800	47.76%	29.32%	0.12%	–	–
0.825	50.61%	32.29%	13.68%	–	–
0.850	53.33%	35.04%	19.11%	–	–
0.875	55.94%	37.61%	22.41%	–	–
0.900	58.44%	40.05%	25.22%	-2.31%	–
0.925	60.85%	42.36%	27.75%	9.07%	–
0.950	63.17%	44.57%	30.07%	16.58%	–
0.975	65.41%	46.68%	32.24%	19.50%	–
1.000	67.57%	48.71%	34.28%	21.96%	-4.71%

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